QUARK MIXING IN CHIRAL $SU_n \times SU_n$ BROKEN SYMMETRY IN THE LIMIT OF EXACT $SU_k \times SU_k$ SYMMETRY

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(Received July 16, 1981)

The structure of chiral $SU_n \times SU_n$ symmetry breaking in the limit of exact chiral $SU_k \times SU_k$ symmetry ($k < n$) is considered. The mixing angle is determined by the masses and decay constants of pseudoscalar mesons.

PACS numbers: 12.40.-y

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Recently, the hierarchy of chiral symmetry breaking has been investigated [1-5]. The symmetry breaking and mixing of quarks are connected with the rotation of quark currents and hamiltonian densities. The determination of the rotation angle becomes an important problem. For the first time the procedure of chiral symmetry breaking, based on the Gell-Mann, Oakes, Renner (GMOR) model [6] has been used in $SU_3 \times SU_3$ symmetry in the limit of exact $SU_2 \times SU_2$ symmetry [1] to determine the value of the Cabibbo angle [7]. The transformation of rotation is connected with the seventh generator of the SU_3 group. After the charmed particles have been discovered the $SU_3 \times SU_3$ symmetry is no longer adequate to describe the strong interactions. The $SU_4 \times SU_4$ symmetry introduced earlier [8] to explain the behavior of charged and neutral currents becomes quite satisfactory model describing the hadron world. The problem of determining the Cabibbo angle in $SU_4 \times SU_4$ symmetry has arisen. It is considered in Ref. [2, 3] and the method of calculating the Cabibbo angle in $SU_4 \times SU_4$ symmetry is described in Ref. [4]. It is known that the formula describing the rotation angle is not changed if the symmetry is extended. This is not unexpected because the Cabibbo angle is connected with the mixing of the d and s quarks and the rotation is performed around the seventh axis in SU_3 subspace too.

The problem of chiral $SU_4 \times SU_4$ symmetry breaking in the limit of exact $SU_2 \times SU_2$ symmetry is considered in Ref. [3]. Symmetry breaking is connected with the transformation of rotation around the tenth axis in SU_4 space. The rotation angle is determined in Ref. [4].

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The other variant of $SU_4 \times SU_4$ symmetry breaking in the limit of exact $SU_3 \times SU_3$ symmetry is described in Ref. [5]. It is connected with the rotation around the fourteenth axis in SU_4 space. In this paper we introduce the general method of rotation angle description in the broken $SU_n \times SU_n$ symmetry. The chiral $SU_n \times SU_n$ symmetry is broken according to the GMOR model. In the first step we introduce the hamiltonian density breaking $SU_n \times SU_n$ symmetry but invariant under $SU_k \times SU_k$ symmetry. In the second step we introduce quark mixing and the resulting exact symmetry is $SU_{k-1} \times SU_{k-1}$.

The particular investigation of cases like the above is not necessary.

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The generalized GMOR model is used. It is assumed that by enlargement to a higher symmetry the new quantum numbers are the charges (as for example: electric charge, strangeness, charm but not isospin). Then the $SU_n \times SU_n$ symmetry breaking hamiltonian density can be written as a linear combination of diagonal operators u^i.

$$H_E = \sum_{j=1}^{n} c_{j^2-1} u^{j^2-1}, \tag{1}$$

where the scalar densities u^i $(= \bar{q}\lambda^i q)$ and pseudoscalar densities v^i $(= i\bar{q}\lambda^i \gamma_5 q)$ satisfy the equal-time commutation rules

$$[Q^i, u^j] = if_{ijk} u^k, \tag{2a}$$

$$[Q^i, v^j] = if_{ijk} v^k, \tag{2b}$$

$$[\bar{Q}^i, u^j] = id_{ijk} v^k, \tag{2c}$$

$$[\bar{Q}^i, v^j] = -id_{ijk} u^k, \tag{2d}$$

where f_{ijk} are the structure constants, d_{ijk} — symmetric constants, $Q^i \pm \bar{Q}^i$ — are the generators of the $SU_n \times SU_n$ group.

If the $SU_k \times SU_k$ symmetry is exact then

$$\partial^\mu V_\mu{}^i = \partial^\mu A_\mu{}^i = 0 \quad (i = 1, \ldots, k^2-1). \tag{3}$$

In the GMOR model the divergences of currents can be calculated as follows

$$\partial^\mu V_\mu{}^i = i[H_E, Q^i], \tag{4a}$$

$$\partial^\mu A_\mu{}^i = i[H_E, \bar{Q}^i]. \tag{4b}$$

We require that the $SU_k \times SU_k$ symmetry be exact, then the following constraints are obeyed

$$c_{j^2-1} = 0 \quad (j = 2, \ldots, k) \tag{5a}$$

$$\sqrt{\frac{2}{n}}\, c_0 + \sum_{j=k+1}^{n} \sqrt{\frac{2}{j(j-1)}}\, c_{j^2-1} = 0. \tag{5b}$$

The symmetry breaking hamiltonian density can be written in the following form

$$H_E = c_0(u^0-\sqrt{n-1}\,u^{n^2-1})+\sum_{j=k+1}^{n-1} c_{j^2-1}\left(u^{j^2-1}-\sqrt{\frac{n(n-1)}{j(j-1)}}\,u^{n^2-1}\right). \tag{6}$$

Using the standard representation of λ matrices one obtains

$$u^0 = \sqrt{\frac{2}{n}}\sum_{j=1}^{n}\bar{q}_j q_j, \tag{7}$$

$$u^{j^2-1} = \sqrt{\frac{2}{j(j-1)}}\left(\sum_{l=1}^{j-1}\bar{q}_l q_l-(j-1)\bar{q}_j q_j\right), \tag{8}$$

$$u^0-\sqrt{n-1}\,u^{n^2-1} = \sqrt{2n}\,\bar{q}_n q_n, \tag{9}$$

$$u^{j^2-1}-\sqrt{\frac{n(n-1)}{j(j-1)}}\,u^{n^2-1} = \sqrt{\frac{2}{j(j-1)}}\left((n-1)\bar{q}_n q_n-j\bar{q}_j q_j-\sum_{l=j+1}^{n-1}\bar{q}_l q_l\right). \tag{10}$$

Let us note that the term $\bar{q}_k q_k$ does not exist in Eq. (6)

$$H_E = \left(\sqrt{2n}\,c_0+(n-1)\sum_{j=k+1}^{n-1} c_{j^2-1}\sqrt{\frac{2}{j(j-1)}}\,\bar{q}_n q_n\right.$$
$$\left.-\sum_{j=k+1}^{n-1} c_{j^2-1}\sqrt{\frac{2}{j(j-1)}}\left(j\cdot\bar{q}_j q_j+\sum_{l=j+1}^{n-1}\bar{q}_l q_l\right)\right). \tag{11}$$

The chiral $SU_n\times SU_n$ symmetry with the exact $SU_k\times SU_k$ subsymmetry is broken by the rotation of the $SU_k\times SU_k$ invariant hamiltonian density around the axis with the index $m = (n-1)^2+2k-1$.

$$H_{SB} = e^{-2i\alpha Q^m}H_E e^{2i\alpha Q^m}. \tag{12}$$

Only the quarks q_k and q_n are mixed. The $SU_k\times SU_k$ symmetry is no longer exact. Only the term $\bar{q}_n q_n$ is rotated under transformation (12), because there is no $\bar{q}_k q_k$ term in the hamiltonian density (11).

$$e^{-2i\alpha Q^m}\bar{q}_n q_n e^{2i\alpha Q^m} = \bar{q}_n q_n-(\bar{q}_n q_n-\bar{q}_k q_k)\sin^2\alpha-\tfrac{1}{2}(\bar{q}_k q_n+\bar{q}_n q_k)\sin 2\alpha. \tag{13}$$

The above consideration is limited to processes not having the change of the quantum number N connected with the SU_n symmetry. So in the broken hamiltonian density

$H_{SB(\Delta N=0)}$ the terms $\bar{q}_n q_k$ and $\bar{q}_k q_n$ do not appear. The broken hamiltonian density is a linear combination of the diagonal operators u^i only.

$$H_{SB(\Delta N=0)} = H_E + A \sum_{j=k}^{n-1} \left(\sqrt{\frac{j+1}{2j}}\, u^{(j+1)^2-1} - \sqrt{\frac{j-1}{2j}}\, u^{j^2-1} \right) \sin^2 \alpha, \tag{14}$$

where

$$A = \sqrt{2n}\, c_0 + (n-1) \sum_{j=k+1}^{n-1} c_{j^2-1} \sqrt{\frac{2}{j(j-1)}}. \tag{15}$$

In more detail equation (14) is given as follows

$$H_{SB(\Delta N=0)} = c_0 u^0 - A \sqrt{\frac{k-1}{2k}} \sin^2 \alpha u^{k^2-1} + \sum_{j=k+1}^{n-1} u^{j^2-1} \left(c_{j^2-1} + A \sin^2 \alpha \sqrt{\frac{1}{2j(j-1)}} \right)$$

$$+ u^{n^2-1} \left(-c_0 \sqrt{n-1} - \sum_{j=k+1}^{n-1} c_{j^2-1} \sqrt{\frac{n(n-1)}{j(j-1)}} + A \sqrt{\frac{n}{2(n-1)}} \sin^2 \alpha \right). \tag{16}$$

If the $SU_k \times SU_k$ symmetry is exact then the pseudoscalar mesons corresponding to the indices $(j = 1, \ldots, k^2-1)$ are massless [9]. After the $SU_k \times SU_k$ has been broken, the $SU_{k-1} \times SU_{k-1}$ symmetry is still exact, because the operator Q^m does not mix the quarks $q_1, \ldots, q_{k-1}$ neither with themselves nor with other quarks. The mesons corresponding to the indices $(j = 1, \ldots, (k-1)^2-1)$ after symmetry breaking are still massless, while the mesons corresponding to the indices $(j = (k-1)^2, \ldots, k^2-1)$ belong to the massive multiplet (k)[1]. The masses of mesons are determined in the GMOR model. Before the $SU_n \times SU_n$ symmetry is broken the masses are described by the coefficients $c_0, \ldots, c_{n^2-1}$ from Eq. (1). After the symmetry has been broken the new factors $c'_0, \ldots, c'_{n^2-1}$ are obtained as the coefficients standing by the operators u^i in the broken hamiltonian density (16) [4]

$$c'_0 = c_0 \tag{17a}$$

$$c'_{k^2-1} = -A \sqrt{\frac{k-1}{2k}} \sin^2 \alpha, \tag{17b}$$

$$c'_{j^2-1} = c_{j^2-1} + A \sin^2 \alpha \sqrt{\frac{1}{2j(j-1)}} (k < j < n), \tag{17c}$$

$$c'_{n^2-1} = -c_0 \sqrt{n-1} - \sum_{j=k+1}^{n-1} c_{j^2-1} \sqrt{\frac{n(n-1)}{j(j-1)}} + A \sqrt{\frac{n}{2(n-1)}} \sin^2 \alpha. \tag{17d}$$

[1] Here "k" is connected with the $SU_k \times SU_k$ symmetry. One should not be misled by the multiplet of K mesons which are connected with the indices $(j = 4, 5, 6, 7)$ [9].

The masses of the mesons are determined as follows [9]

$$m_{a}^{2} f_{a}^{2} \delta_{ab} = \sqrt{\frac{2}{n}} \left(c_0 d_{0ab} + \sum_{j=k+1}^{n} c'_{j^2-1} d_{j^2-1ab} \right) (u^0)_0. \tag{18}$$

The relation between the indices a, b, j and meson states is described, for example, for the $SU_4 \times SU_4$ symmetry in Ref. [9, 10]. For $a = b = k^2-1$, the mass of the (k) meson is given as follows

$$m_{(k)}^2 f_{(k)}^2 = \sqrt{\frac{2}{n}} \left(\sqrt{\frac{2}{n}}\, c_0 + \sum_{j=k+1}^{n} d_{j^2-1ab} c'_{j^2-1} \right) (u^0)_0 = \frac{1}{k} \sqrt{\frac{2}{n}}\, A \sin^2 \alpha (u^0)_0. \tag{19}$$

For $a = b = m = (n-1)^2+2k-1$, the mass of the (n) meson is given by

$$m_{(n)}^2 f_{(n)}^2 = \sqrt{\frac{2}{n}} \left(\sqrt{\frac{2}{n}}\, c_0 + \sum_{j=k+1}^{n} d_{j^2-1mm} c'_{j^2-1} \right) (u^0)_0. \tag{20}$$

Because

$$d_{j^2-1mm} = \sqrt{\frac{1}{2j(j-1)}} \quad (j < n), \tag{21}$$

$$d_{n^2-1mm} = \frac{2-n}{\sqrt{2n(n-1)}}, \tag{22}$$

so

$$m_{(n)}^2 f_{(n)}^2 = \tfrac{1}{2} \sqrt{\frac{2}{n}}\, A \left(1 - \left(1 - \frac{1}{k} \right) \sin^2 \alpha \right) (u^0)_0. \tag{23}$$

In formulae (19) and (23) to determine the masses of (k) and (n) mesons one has $(n-k+3)$ unknown quantities with which to deal $(c_0, c_{(k+1)^2-1}, \ldots, c_{n^2-1}, (u^0)_0, \sin \alpha)$. Nevertheless the angle α is determined by the masses and decay constants of two pseudoscalar mesons (k) and (n) only.

$$\sin^2 \alpha = \frac{k \cdot m_{(k)}^2 f_{(k)}^2}{2m_{(n)}^2 f_{(n)}^2 + (k-1) m_{(k)}^2 f_{(k)}^2}. \tag{24}$$

All the cases of symmetry breaking considered in Refs [1–5] can be described by formula (24). Let us give simple examples:

a) for $k = 2$, $n = 3$ α is the original Cabibbo angle θ associated with rotation around the seventh axis in SU_3 subspace [1, 4]

$$\sin^2 \theta = \frac{2m_\pi^2 f_\pi^2}{2m_K^2 f_K^2 + m_\pi^2 f_\pi^2}, \tag{25}$$

b) for $k = 2$, $n = 4$ and rotation around the tenth axis [3, 4] one obtains

$$\sin^2 \alpha = \frac{2m_\pi^2 f_\pi^2}{2m_D^2 f_D^2 + m_\pi^2 f_\pi^2}, \tag{26}$$

c) for $k = 3$, $n = 4$ and rotation around the fourteenth axis [4, 5] one obtains

$$\sin^2 \alpha = \frac{3m_K^2 f_K^2}{2(m_D^2 f_D^2 + m_K^2 f_K^2)}, \tag{27}$$

d) for $k = 3$, $n = 5$ and rotation around the twenty first axis [11] one obtains

$$\sin^2 \alpha = \frac{3m_K^2 f_K^2}{2(m_G^2 f_G^2 + m_K^2 f_K^2)}. \tag{28}$$

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In general the determination of the rotation angle (24) in $SU_n \times SU_n$ symmetry is possible only if the new quantum numbers introduced by a transition to the higher symmetry are scalars of the charge type (additiv). So the hamiltonian density (1) can be constructed as a linear combination of the diagonal operators u^i only ($i = j^2-1, j = 1, ..., n$). The method of determining the rotation angle, discussing and interpreting the symmetry breaking is described in more detail in Ref. [4] on $SU_4 \times SU_4$ symmetry as an example.

The author would like to thank Professor W. Tybor for helpful discussions and remarks.

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