On possibilities of the fluorescence detector to measure the shower light curve

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Received 26 March 2002; received in revised form 11 May 2002; accepted 11 May 2002

Abstract

A method to determine the primary energy of very rare big extensive air showers is to measure the fluorescence light flashes induced by them in the atmosphere. From a shower fluorescence image (and its time dependence) it is, in principle, possible to reconstruct the shower cascade curve. The Pierre Auger experiment (in construction) has been using this method (together with measuring the shower charged particles as well) to determine the highest energy part of the cosmic ray spectrum \( E \gtrsim 10^{19} \text{ eV} \) and particle arrival directions.

Here we analyse which shower parameters affect its image, and, if not taken into account in the reconstruction procedure, may lead to systematic errors in determining its light (cascade) curve, and in consequence, the energy and/or mass of the primary particle. In particular, we analyse the lateral distribution of particles, the thickness and curvature of the shower disk, together with the spherical aberration of the collecting mirror. We show that a non-negligible part of the light flux for showers closer than \( \sim 15 \text{ km} \) to the detector may be hidden in the non-triggered pixels of the camera. For more distant showers this effect is small, but then the atmospheric attenuation has to be known better.

We also derive an analytical solution for a spherical mirror focal plane position and the minimal size of the image (spot size) of a parallel light beam.

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1. Introduction

The problem of the origin of the ultra high energy \( (E>10^{19} \text{ eV}) \) cosmic rays (CR) has recently been tackled by many authors. It is because the information about sites of their origin, in contrast to the lower energy particles, may hopefully be deduced from their energy spectrum and the arrival direction distribution. To be more specific: calculations show that if protons with \( E > 10^{19} \text{ eV} \) were produced in the Galactic disk, their arrival directions would be correlated with it (which, however, may not be true if CR were iron nuclei and/or there was a large (\( \gtrsim 20 \text{ kpc} \)) magnetic halo in the Galaxy [1]). Protons and nuclei of such a high energy are also prone to the propagation in the intergalactic space. The existence of the cosmic microwave and infrared backgrounds limits distances to the particle sources only to several tens of Mpc (very well known as the Greisen–Zatsepin–Kuzmin (GZK) effect [2]). Moreover, it seems that

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the extragalactic irregular magnetic fields are low enough not to deflect particle trajectories significantly, so that with a bigger statistics the arrival directions may show us where the particles come from.

The experiment just devoted to increase significantly the number of registered extensive air showers above $10^{19}$ eV is the Pierre Auger Observatory (PAO) [3] in construction. Its aim is to determine arrival directions and energy spectrum of CR in the GZK region and verify the intriguing observation of the non-existence of the GZK cut-off [4]. The energy determination of an extensive air shower, usually made on the basis of the total number of charged particles detected on the Earth, is not a very accurate method as its predictions depend rather strongly on the nuclear interaction model adopted, about which nothing certain can be said at such high energies. However, the Auger experiment, apart from the particle detectors on earth, will measure the fluorescence light emitted by the atmosphere during the shower development in it. This method was applied for the first time by the Fly's Eye experiment [5] but without particle detectors. Fluorescence light has a big advantage of being emitted isotropically (in contrast to the Cherenkov light, which is collimated with the shower direction) resulting in a large effective area of its detection. Moreover, its flux from a given direction is approximately proportional to the number of charged particles within the field to view. Thus, observing the angular distribution of the fluorescence light from a shower, enables (in principle) to determine the cascade curve, i.e. the number of particles not only on earth but as a function of height above it. If the atmospheric attenuation is known it is then possible to determine the total fraction of the primary energy transferred to the electromagnetic component of the shower. The contribution of muons and hadrons can be estimated from Monte Carlo shower simulations. The primary energy is obtained by adding up energies of all the components. Accurate energy determination is particularly important in the region where one expects some changes of the spectrum shape (as in the case of the problematic GZK cut-off) because the steepness of the spectrum together with large energy uncertainties would smear and shift to higher values the cut-off energy.

Another important goal of the Auger experiment is the estimation of the primary particle masses. The most reliable method is the study of atmospheric depths of shower maxima and their fluctuations for a fixed primary energy, by measuring fluorescence images of individual showers. Independently of the nuclear interaction model the shower maximum depth and particularly its fluctuations from shower to shower should be smaller for heavy nuclei than those for protons.

The hybrid method of the Auger will allow for the first time to compare parameters of the same shower, determined by the surface array, with those from the fluorescence detector (FD). A continuous monitoring of the atmosphere attenuation will allow to determine the absolute light fluxes emitted and, consequently, to calibrate the primary energy scale for the surface array. (Note that the majority of events will be registered by the surface array only.)

In this paper we analyse some possibilities of the Auger FD to measure the angular distribution of light (images) registered from individual showers. The image of a shower, integrated over time, is a narrow track on the sky. In evaluating its detection possibilities it is often assumed that it is an infinitely thin line. This overestimates the predicted number of shower photons above the night sky background. Here we consider several factors, describing showers in more detail and analyse conditions when they are important. These are the lateral distribution of shower particles broadening the light track and the time pulse from a PMT, the curvature of the shower front changing the time difference between signals from adjacent directions (PMTs), and finally, the spherical aberration of the collecting mirror, blurring the image.

2. Detection of the fluorescence light by the FD camera

The PAO [3] is being built in the southern hemisphere in Argentina. It consists of two sorts of detectors:
the so-called “surface detector” (SD) for shower particle detection and
the FD for detection of the fluorescence light emitted by atmospheric molecules of nitrogen, excited by charged particles of the shower.

The surface covered by particle detectors (~1500 large water tanks, separated by 1.5 km) will be ~3000 km². Four FDs will be situated on the borders of this area to collect light from the showers registered by the SDs. Each FD (eye) consists of six adjacent telescopes, each having a field of view 30° in azimuth and 30° in elevation angle (1–31°) looking towards the surface array. Each telescope collects light falling on a spherical mirror (3 m² of effective area) focusing it on a spherical camera, situated in the mirror’s focal plane. The camera consists of 440 hexagonal photomultipliers (PMTs), each having 1.5° field of view. The light signal from a PMT is integrated in 100 ns intervals. Thus, the angular and time resolutions of the shower image are ≤ 1.5° and 100 ns respectively.

The most important goal of the FD is to measure the whole fluorescence light falling on the telescope diaphragm. Here we shall concentrate on the camera itself and study which approximations about the shower phenomenon are allowed (or not), in order to correctly determine the total number of fluorescence photons, as a function of height and time, induced by shower particles.

3. Shower track width integrated over time

First, we shall deal with the lateral distribution of shower particles and study the correctness of neglecting it in the process of reconstructing the shower curve from the fired PMTs (pixels).

The lateral spread of the shower particles causes a finite width of a shower image. It has been determined by many experiments at lower energies (~10¹⁵–10¹⁶ eV) [6] and it is usually described in terms of the Nishimura–Kamata–Greisen (NKG) function:

\[
f(r) \sim \left( \frac{r}{r_M} \right)^{s-2} \left( 1 + \frac{r}{r_M} \right)^{s-4.5}
\]

where \( s \) is the so-called shower age parameter and \( r_M \) is the Molière radius, defined as

\[
r_M = \frac{21 \text{ MeV}}{E_{cr}} x_0
\]

where \( E_{cr} \) is the critical energy (~80 MeV), and \( x_0 \)—the cascade unit (~38 g cm⁻²) of the air.

From the pure electromagnetic cascade solution it follows that \( s \) increases with the cascade development and \( s = 1 \) for the lateral distribution at the cascade maximum. The experiments with extensive air showers, which are a superposition of many electromagnetic cascades initiated at various atmospheric depths with various energies, actually show that the description of showers in the mentioned energy region in terms of the age parameter \( s \) is rather good.

It is very difficult to measure the lateral spread of particles in very big showers close to their axes with reasonable costs, because such showers are very rare and the required large areas have to be covered by detectors, of necessity with large spacing. However, the lateral distribution of particles in large showers has been measured by the Haverah Park experiment [7]. The measurements for \( 2 \times 10^{17} < E < 4 \times 10^{18} \) eV concern the distance region \( r > 50 \) m.

For \( E > 10^{19} \) eV the exact shape for the bulk of shower particles, i.e. for \( r < 100 \) m, can only be obtained from extrapolation from larger distances. These measurements concern sea level, but here we are interested in the shower maximum region. Firstly, because the particle number there is proportional to the primary particle energy; secondly, the height of the maximum, as explained in the Introduction, is a measure of the primary particle mass. Thus, we shall study shower images around their maxima. Therefore, one has to rely on shower simulation results.

The calculations done with the use of the CORSIKA program [8] show that for \( 10^{19} \) eV proton showers the lateral distribution function at the shower maximum close to the core is described by \( s = 0.8 \) rather than by \( s = 1 \). As its behaviour in the shower maximum should depend rather weakly on primary particle energy and mass, we shall assume in what follows that \( s = 0.8 \) corresponds to shower maxima for all energies above
10^{19} \text{ eV} \text{ and primary heavy nuclei as well as protons.}

If the image of the shower axis passes through the center of a pixel (central pixel) the number of particles seen (being proportional to the number of collected photons) is that contained within a band centered on the axis, with its width equal to the product of the pixel angular size (1.5°) and distance to the shower (Fig. 1). Here we have assumed that the shower’s distance from the detector is large enough, so that the two pixel field of view limiting lines are parallel.

Since the NKG function for a given \( s \) depends on the dimensionless distance \( x = r/r_M \) only, one can calculate a universal curve giving the fraction of the total number of particles seen by the central pixel. The results are presented in Fig. 2. It can be seen that within the band \( 2r_M \) (\( x = 1 \)) there are <90% of particles for \( s > 1 \). For \( s = 0.8 \) half of them are contained within \( 2 \times 0.1r_M \). The dashed vertical lines correspond to pixel fields of view for showers at different distances \( l \). They are drawn for \( r_M = 140 \text{ m} \), the value corresponding to the height in the atmosphere where proton initiated showers with \( E_0 = 10^{20} \text{ eV} \) and zenith angle 45° have their maxima. For \( s = 1 \) and distances 5, 10 and 20 km the fractions outside the central pixel are 19%, 7% and 2% respectively. Below the maximum they become larger.

It is worth noticing that these results do not depend on the angle \( \theta \) between the line of sight and the shower axis, as long as this angle is not too small. To be precise, the calculations have been performed under the assumption that all shower parts seen by a pixel are at the same stage of development (the same \( s \)). This is not quite true but as most of particles are contained within \( r \approx r_M \), i.e. in a region small compared to the longitudinal scale of shower development, the difference of \( s \) between closer and more distant shower parts is small as well.

In the above calculations we have neglected the fact that the pixel field of view is a hexagon, that is the track segment passing through its center is longer than that at its side. The present calculations are supposed to be some estimations of several effects rather than very accurate predictions. Anyway, the exact fractions of a signal in individual pixels would depend on a particular position of the shower line (image of the shower axis) on the hexagon’s grid.

So far, we have also neglected the spherical aberration of the mirror. This effect causes a parallel beam of light incident on the spherical mirror to form a spot of a finite size on the focal surface. An analytical solution for the spot size has been found by us (see Appendix A). For the Auger FD

Fig. 1. Field of view of the central pixel contains particles within band of width 2\( x \).

Fig. 2. Fraction of particles contained within 2\( x \) for the NKG lateral distribution for several values of \( s \). Dashed vertical lines correspond to fields of view of the central pixel for showers 5, 10, 20, and 30 km away, for \( r_M = 140 \text{ m} \). Spherical aberration is not yet taken into account.
the diameter of the spot comes out to be 0.5°, coinciding with the value obtained earlier numerically [9].

To take into account the spherical aberration we shall assume that the light intensity across the spot is uniform (for a constant intensity in the incident beam). It is not quite true for a perfect spherical mirror [10], but real mirror imperfections smooth to some extent the intensity profile. Anyway, as the effect of aberration is rather small, what will be shown later, this is a reasonable simplification.

As the spherical aberration blurs the image, allowing for it is equivalent to modifying (blurring) the lateral distribution of particles and leaving the mirror without aberration. The modified lateral distribution function \( f'(r) \) can then be calculated from the formula:

\[
 f'(r) = \frac{\int f(r') \, ds(r')}{\pi \rho^2} \tag{3}
\]

where \( \rho \) is the blurring radius, equal to 0.25° × distance to the shower, \( r' \) is the core distance and \( ds(r') \) is the surface of the ring element \( (r', r' + dr') \) contained within the circle of radius \( r \) with the center at core distance \( r \). The integration is carried out over the whole surface of this circle.

Fig. 3 shows the influence of this effect on the fraction of light seen by the central pixel. Here, similarly to Fig. 2, there are drawn fractions of the signal, registered in the central pixel (for \( s = 0.8 \)) but obtained from \( f'(x) \), for distances \( l = 5, 10, 20 \) and 30 km. Fields of view of the pixel are also shown by vertical dashed lines. The dashed upper curve corresponds to the NKG function \( f(x) \) (with no aberration taken into account). The curves with and without aberration differ significantly for small \( x \) for all distances meaning that the distribution of light within the pixel is changed. For distances larger than 20 km, the aberration affects only slightly the number of particles seen by the central pixel. For smaller distances, however, its effect obviously increases: for 5 km the fraction falls from 89% to 64%, and for 10 km—from 97% to 88%. For the shower levels above the maximum (\( s < 0.8 \)) the aberration effect will be relatively stronger.

4. Time dependence of the light signals

Now we shall investigate the time evolution of the light flux arriving at a particular pixel. The instantaneous image of a shower has circular symmetry resulting from the axially symmetric lateral distribution of particles [11]. If the angle between the shower axis and the line of sight is \( \theta \), then photons arriving simultaneously at a pixel would have been produced by particles while crossing a plane inclined at angle \( \theta/2 \) to the shower axis, e.g. C’D’ or F’E’ in Fig. 4. As time progresses the plane moves away from the detector. In other words, a pixel sees the light from the plane moving away from the detector with the light velocity \( c \) along the line of sight. Thus, the time profile of the light flux falling on a given pixel reflects directly the variation of the number of particles on that plane ‘cut-out’ by the pixel’s field of view (Fig. 5).

Fig. 6 represents the results of calculations for \( s = 0.8 \) and \( s = 1 \). Time profiles of the light flux are shown for pixels seeing a shower perpendicularly to the axis, from three different distances \( l = 5, 10 \) and 20 km. On each graph there are two sets of curves corresponding to the two pixels 1 and 2 (Fig. 5). Thus the angular distance of the center of pixel 2 from the shower axis at time \( t = 0 \) is 1.5°. For each pixel there are drawn three curves.
The solid line describes the signal time profile for an infinitely thin shower disk (a plane) with no spherical aberration taken into account. The dotted line allows for the aberration. Additionally, considering a finite thickness of the shower disk, we obtain the dashed curve.

Fig. 4. Shower detector plane when the shower disk is at some position AB at time $t_0$. Photons being at this instant on line CD have been produced earlier on line C'D' which is a bisector of $\angle CAB$. Similarly, photons being at a later time on line FE have been produced on line F'E' (bisector of $\angle F'A'B$). However, photons at $C'$ have been produced later than those at $F'$ by the time $CF \cos \theta/c$, where $\theta$ is the angle between line of sight and the shower axis. Therefore the arrival time interval $\Delta t_0$ of the considered slab of photons CDEF equals $\Delta t_0 = CF/c = CF(1 - \cos \theta)/c = \Delta t(1 - \cos \theta)$ where $\Delta t$ is the arrival time interval for the shower observed perpendicularly to its axis ($\theta = 90^\circ$).

To allow for the last effect we assumed that particle density at a given instant changes as a Gaussian function along the line perpendicular to the disk, with $\sigma = 30$ m (100 ns), the modified NKG density at this distance being the integral over the Gaussian distribution. Such a shower disk is just a sum of many thin disks entering pixel's field of view with increasing time delays. The actual numbers on the vertical axis are the fractions of all shower particles at a given level seen by the pixel at a given time. Time $t = 0$ corresponds to the shower core crossing the center of pixel 1.

Inspection of the graphs in Fig. 6 leads to the following conclusions:

- The signal maximum in pixel 2 can reach several percent of that in pixel 1, if the distance $l$ to the shower is smaller than 10 km. However, as the shower line would go close to a pixel center rather rarely, this fraction should be treated as a lower limit.
- The spherical aberration becomes important for $l > 10$ km; for $s = 0.8$ the effect is always more significant than that for $s = 1$ because of a steeper lateral distribution for the former case. The smoothing of the trapezoidal shape is signifi-
The effect of a finite disk thickness can only be important for steep signal slopes, i.e. for small distances. It is not clear what this thickness is.

cant for \( l \geq 20 \) km; for smaller distances the shapes are far from trapezoidal (an assumption made quite often).
for big showers near the axes, but our calculations show that this effect would be important for \( l \leq 5 \) km only, if \( \sigma \geq 30 \) m.

The shape of the signal in the side pixel 2 could actually be deduced from that in the central one, due to the circular symmetry of the disk image, when the shower axis crosses the center of pixel 1. For example, the signal maximum in pixel 2 should be equal to the value corresponding to \( t = \pm 1.5^\circ \times l/c \) for pixel 1 (we assumed that the pixel field of view is circular, which is almost true, with the solid angle the same as that for the hexagon with 1.5° side to side). The time scale in Fig. 6 is appropriate for the line of sight perpendicular to the shower axis (\( \theta = 90^\circ \)). For another angle \( \theta \) one has to multiply the time scale by \( 1 - \cos \theta \) (see caption to Fig. 4).

5. Absolute light fluxes

The results presented so far do not depend on the primary energy of a shower, if the shape of the lateral distribution function depends on the age parameter \( s \) only. (Actually, it does depend to some extent on the primary energy because the most representative value of the Moliere radius \( r_M \) (in meters) should be larger for smaller energies.) However, we would like to investigate now how much of the light flux would be hidden in the side pixels which have not been fired and this, of course, depends on absolute light fluxes and, therefore, on primary energy.

Each charged particle causes an isotropic emission of 4.5 fluorescence photons per 1 m of its track. For the attenuation length of light in the atmosphere we adopt \( \lambda = 8.4 \) km and the rest of the necessary parameters are appropriate for the FD: mirror effective area—3 m\(^2\), its reflectivity—0.9, photocathode efficiency—0.3.

At the Auger site in Argentina the background corresponds to about 58 photoelectrons (pe) emitted from the PMT cathode per 1 \( \mu \)s. A pixel is triggered if the number of photoelectrons in 10 consecutive 100 ns time bins exceeds, in an adopted way, the average background in 1 \( \mu \)s. The Auger Collaboration has recommended that an excess of at least four standard deviations should trigger a pixel. (The FD trigger for a shower requires five adjacent pixels to be fired.)

For an adopted background light one can find the minimum distance \( l \) to the shower level when the side pixels remain inactive. Fig. 7 shows the calculated distances to the shower level \( s = 1 \) as a function of the primary energy (solid lines—primary protons, dashed—iron), for which the signal in both side pixels equals to the fraction indicated (falling lines). The rising lines show the distance above which the side pixel would not be triggered, for two background values: 58 pe/\( \mu \)s (upper line) and 4 \( \times \) 58 pe/\( \mu \)s (primary proton).

![Fig. 7. Distance to the shower level \( s = 1 \) as function of the primary energy ((—) primary proton, (---) iron), above which signal in both side pixels is smaller than the fraction indicated (falling lines). The rising lines show the distance above which the side pixel would not be triggered, for two background values: 58 pe/\( \mu \)s (upper curve) and 4 \( \times \) 58 pe/\( \mu \)s (primary proton).](image-url)
6. Shower disk curvature and its image

A question may arise to what extent a curvature of the shower disk can change its instantaneous image, i.e. our results obtained so far. Of course, this effect should not be as important for the fluorescent image of a shower as it is for its ground characteristics, where arrival times of particles are measured over distances of several kilometers.

Here we are interested in the curvature effect up to the core distances of \( r \leq 200 \) m or so, containing the bulk of particles. The situation is illustrated in Fig. 8. We have assumed that the shower disk is a thin plane. For its particular shape we have taken simulation results of Sciutto [12] for the electron mean delay time as a function of the distance from the shower core, for \( E = 10^{20.5} \) eV (primary proton). It has been drawn in scale as a solid line in Fig. 8 where the ordinate coincides with the shower axis. When its image crosses the center of a pixel, the distance ranges seen from 10 km by the central and both side pixels would be those marked by the vertical dashed lines. For this particular configuration the particles seen by the central pixel would be delayed by \( \leq 50 \) ns with respect to those on the axis. We have additionally drawn two curves from Fig. 2 showing the fraction \( F \) of all particles on levels \( s = 0.8 \) and 1 within band \( 2r \). The time spread due to the disk curvature in the side pixels is bigger (\( \sim 200 \) ns) but only about 3% of all particles for \( s = 1 \) would be delayed by more than 100 ns.

7. Conclusions

In this paper we have investigated to what extent the characteristics of real showers, such as lateral spread of particles, finite disk thickness and its curvature, would affect their images. Assuming that the particle lateral distribution is described well by the NKG function we have shown that side pixels may contain non-negligible fractions of the light flux, without being triggered. The actual numbers depend, of course, on the background level adopted, but it seems that it is important to look for the signal in the non-triggered side pixels as well in the shower reconstruction procedure.

Time profiles of the signals are not well described by trapezoidal shapes, as it is often considered: for small distances the signals are too short, for large distances—the spherical aberration smoothes the shapes.

A finite disk thickness smoothes slightly the time profiles of the signals, this effect dominating that of the aberration for \( l \leq 15 \) km (if \( \sigma = 30 \) m).

Due to shower disk curvature particles distant by more than \( \sim 200 \) m from the axis (near shower maximum) would be delayed by more than 100 ns, their fraction being however a few percent only.
We have also found an analytical expression for the position of the focal plane and the spot size.

Acknowledgements

This work has been supported by the Polish Research Committee (KBN) grant no. 2 P03C 006 18. The authors thank Z. Szadkowski for useful discussions.

Appendix A. Analytical solution for the spot size of a spherical mirror

It is well known that a parallel beam of light, reflected from a perfect, spherical, concave mirror, is not focused in one point. The rays closer to the mirror axis cross it at a larger distance from the mirror, approaching the value \( R/2 \), where \( R \) is the mirror radius. Thus, at a screen perpendicular to the axis, placed at some distance from the mirror, a parallel incident beam will form a spot of a finite size. There is, however, one such position on the screen (focal plane) when the spot size is the smallest.

We have not found in the literature any analytical derivation for the focal plane position, nor for the spot size. (The practical way of finding these values is, of course, using computers.)

Here, we present an analytical solution of this problem assuming that the diaphragm radius is smaller than that of the mirror (more precisely—see later). Fig. 9 presents the path of a ray distant by \( h \) from the mirror axis OA. It crosses the axis at a distance \( r \) from the center of the mirror. The screen is situated at a distance \( r_0 \). After being reflected, the ray falls on the screen at a distance \( a \) from the axis. From the figure geometry it can be derived how \( a \) depends on \( h \) (in fact, all relations depend on ratios of any segment length to the mirror radius \( R \)):

\[
\frac{a}{R} = (\rho \sqrt{1 - z - 1}) \frac{\sqrt{z}}{1 - 2z}
\]  

(A.1)

where we have introduced the notations \( z = (h/R)^2 \) and \( \rho = 2r_0/R \).

From the above formula, as well as from an insight into the reflected ray paths we can deduce a qualitative behaviour of \( a \) as a function of \( h \), for some screen position \( r_0 \). Fig. 10 shows roughly this dependence for three values of \( r_0 \). If the screen is situated at \( r_0 = R/2 \) (the focus for \( h \to 0 \)) then if \( h \) increases then \( |a| \) will increase as well (negative \( a \) means that the ray falls on the screen below the axis). For \( r_0 > R/2 \) there are two \( h \) values, where \( a = 0 \) (see (A.1)): one for \( h = 0 \) and another for such \( h \neq 0 \) when the reflected ray falls on the screen at its crossing with the axis. For intermediate values (\( 0 < h < h_1 \)) \( a > 0 \), and if the diaphragm radius \( d > h_1 \) \( a \) becomes negative for
\[ h > h_1, \text{ reaching its biggest } |a| \text{ value for } h = d. \text{ The maximum value of } |a(h)| \text{ for a given screen position } r_0 \text{ defines the spot size.} \]

Our method is based on noticing the following. As \( r_0 \) increases (Fig. 10) \( a_{max} \) increases and \( |a(d)| \) decreases. Thus, the smallest spot size will be for such \( r_0 \) when the two values are equal.

First, we shall find \( a_{max} \) as a function of \( r_0 \). Differentiating \( a \) (A.1) with respect to \( z \) and putting \( \delta a/\delta z = 0 \) we obtain

\[
z_m^3 - \frac{3}{4} z_m + \frac{\rho^2 - 1}{4} = 0 \quad \text{(A.2)}
\]

where \( z_m = (h_m/R)^2 \) and \( h_m \) is defined by \( a(h_m) = a_{max} \). The solution of (A.2) is

\[
z_m = \cos \left[ \frac{1}{3} (\phi + 2k\pi) \right] \quad k = 0, 1, 2, \ldots \quad \text{(A.3)}
\]

where \( \cos \phi = 1 - \rho^2 \).

For \( \rho \to 1 \) we must have \( z_m \to 0 \), hence \( k = 0 \) and \( \phi = (3/2)\pi \).

Thus, from the continuity of the solution we must have \( k = 0 \) for \( \rho \geq 1 \) as well. Substituting \( z_m = \cos(\phi/3) \) to (A.1) we find \( a_{max} \).

Now, the best screen position \( r_0 \) can be found by equating \( a_{max} = |a(d)| \):

\[
(\rho \sqrt{1 - z_m} - 1) \frac{\sqrt{z_m}}{1 - 2z_m} = \left( \rho \sqrt{1 - \delta} - 1 \right) \frac{\sqrt{\delta}}{1 - 2\delta}
\]

(A.4)

where \( \delta = (d/R)^2 \) is a fixed value.

At this point we have to make some approximation to unfold (A.4) (note that \( z_m \) depends on \( \rho \)). Thus, we assume that \( \rho^2 - 1 \ll 1 \), meaning that the best screen position is close to the “focus” at \( r_0 = R/2 \). From (A.3) we obtain that

\[
z_m \approx \frac{1}{3}(\rho^2 - 1) \quad \text{(A.5)}
\]

Taking this to the account and assuming further that \( h_m < R \), so that \( (h_m/R)^4 \ll 1 \) and can be neglected, we get a third order equation for \( \sqrt{z_m} \). Solving it and finding \( \rho \) from (A.5) we finally obtain that the best position \( \rho \) equals

\[
\rho = \sqrt{1 + 3x^2} \quad \text{(A.6)}
\]

where

\[
x = A \left[ \cos \left( \frac{\phi}{3} \right) - \frac{1}{2} \right]
\]

where

\[
\cos \phi = - \left( 1 + 4 \frac{A - B}{A^3} \right)
\]

where \( A = B\sqrt{1 - \delta} \) and \( B = \sqrt{\delta}/(1 - 2\delta) \).

The minimum spot size is obtained, as it has been explained, by inserting \( \rho \) calculated from (A.6) to the expression (A.1), with \( h = d \) (or \( h = h_m \)).

For the Auger mirror, \( R = 3.4 \text{ m and } d = 0.85 \text{ m} \). Our solution gives for the best camera (screen) position \( r_0 = \rho(R/2) = 1.742 \text{ m and for the corresponding spot angular diameter } 0.48^\circ \). The accurate computer “ray tracing” calculations (performed by several Auger collaborators) give \( r_0 = 1.743 \text{ m and } 0.50^\circ \) respectively. Thus, our approximations have been justified for this case.

References